Numerical approximation of Stochastic Partial Differential Equations with singular drift

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Areas : Probability, Numerical analysis.

Context : Many physical situations involve diffusion phenomena which are perturbed by random fluctuations (obviously in physics, but also in biology, finance, etc.). In this proposal, we focus on the numerical approximation of some nonlinear parabolic PDEs perturbed by an additive space-time white noise (see [4] for examples and applications). As can be seen from the recent monograph on the numerical analysis of SPDEs [7], only the case of Lipschitz coefficients is well-understood so far.

When the Lipschitz property is lost, particular care is required, as counter-examples show that a lot of regularity is lost even when the coefficients are smooth (see [5] where examples of numerical schemes for SDEs with no possible rate are constructed).

In the case of of smooth but non-globally Lipschitz coefficients, ... for instance one-sided Lipschitz [2] and references therein

PhD proposal : The main aim of this thesis is to establish convergent numerical schemes for SPDEs with irregular drift b, which can be a non-Lipschitz function or even a distribution. We will for instance consider the stochastic heat equation :

$$du_t = \Delta u_t \, dt + b(u_t) \, dt + dW_t, \tag{1}$$

where W is a space-time white noise. As a toy example, we consider this equation in one space dimension with a reflection term, which appears in the equation as a Dirac mass at 0 ($b(x) = \delta_0(x)$). The well-posedness of this equation has been proven very recently [1], for more general drifts in some Besov spaces. The corresponding finite difference scheme with Euler approximation in time reads :

$$U_{n+1}^{(\Delta t)} = S_{\Delta t} \left(U_n^{(\Delta t)} + b(U_n^{(\Delta t)})\Delta t + \Delta W_n \right), \tag{2}$$

where $S_{\Delta t} = (I - A\Delta t)^{-1}$, A is the discrete Laplacean matrix for a mesh size Δx in space and $\{U_n^{(\Delta t)}, n = 0, \dots, \frac{T}{\Delta t}\}$ is thus a multidimensional Euler scheme.

As a first step, the student will work on Euler schemes such as (2) in the case where b is a Hölder function (see e.g. previous works [6] and references therein) and will then study the convergence as $\Delta x \to 0$ and $\Delta t \to 0$.

Then, a new approach will be investigated : it will concern the study of Euler schemes for multidimensional SDEs with distributional (Besov) drifts, using regularisation by noise techniques and averaging operators [3]. Such results would be new even in the case of finite-dimensional SDEs.

Finally, the student will study how the previous approach extends to the infinite-dimensional setting of SPDEs, in order to prove the convergence of (2) to (1), hence providing a good numerical scheme. More broadly, this approach could potentially be extended to other SPDEs and provide new well-posedness results for SPDEs with singular drift.

Références

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